CSE 250
Data Structures

Dr. Eric Mikida
epmikida@buffalo.edu

Dr. Oliver Kennedy
okennedy@buffalo.edu

212 Capen Hall

Day 13
QuickSort and Average Runtime
Textbook Ch. 15

Announcements

- WA1 due tonight at 11:59PM
  - Late submissions (up to tomorrow at 11:59PM) receive 50% penalty
- PA2 is released
  - Start early……please :)

Recap - Merge Sort

Divide: Split the sequence in half

$D(n) = \Theta(n)$ (can do in $\Theta(1)$)

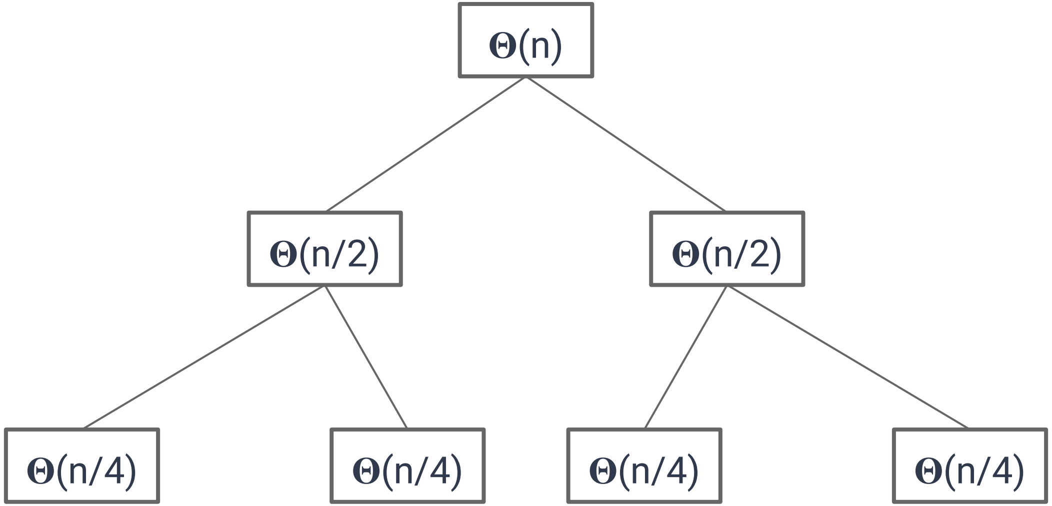
Conquer: Sort the left and right halves

$a = 2, b = 2, c = 1$

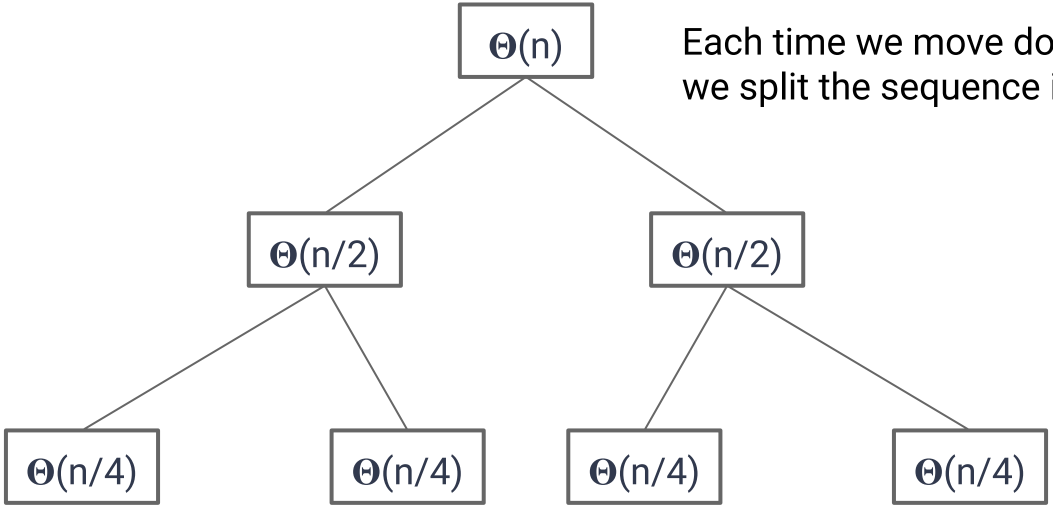
Combine: Merge halves together

$C(n) = \Theta(n)$

Merge Sort: Intuition



Merge Sort: Intuition



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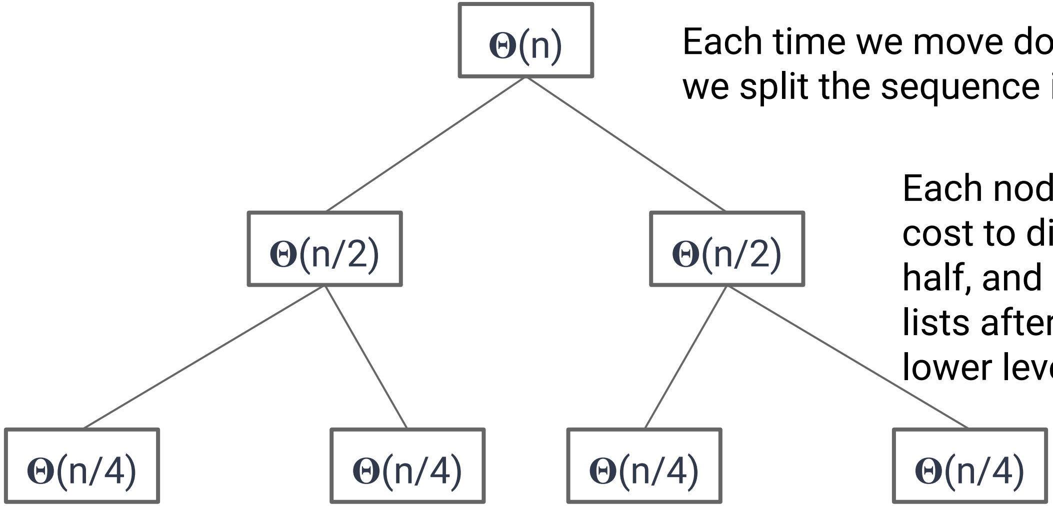

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Each node is labeled with the total cost to dividing the sequence in half, and combining the sorted lists after they are sorted by the lower levels

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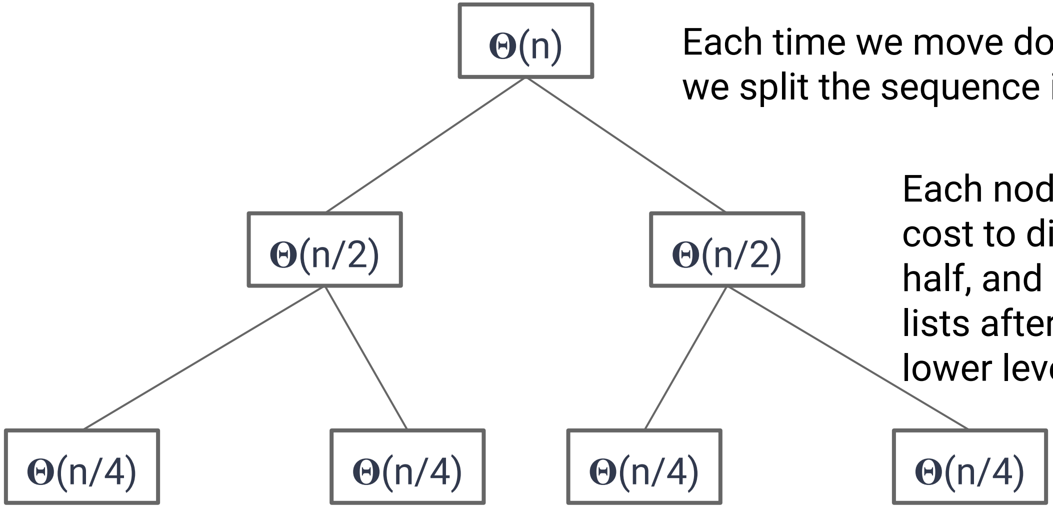

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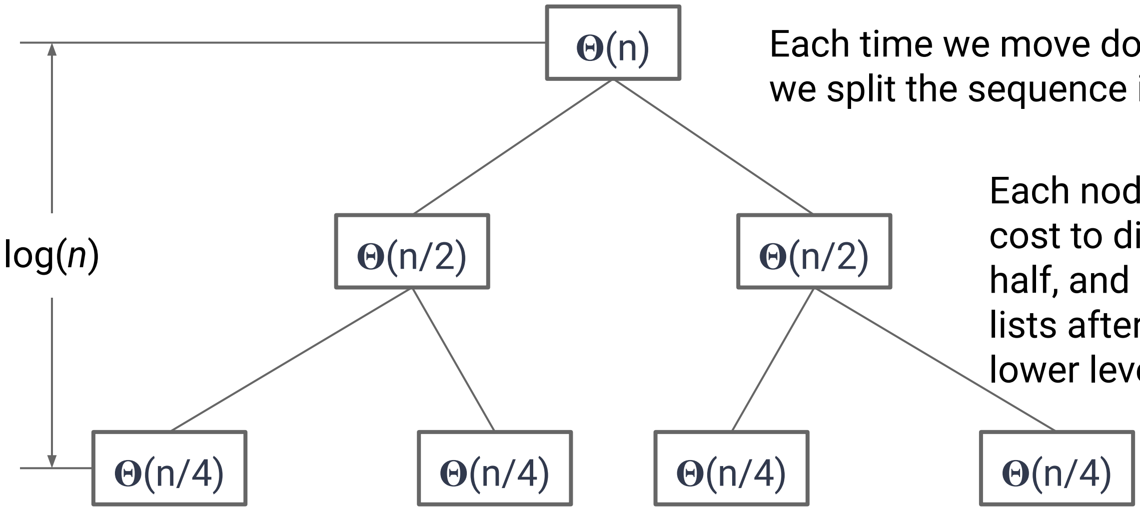
Notice the total cost of each level is always $\Theta(n)$

Merge Sort: Intuition

Because we divide in half at each level, we have log(n) levels



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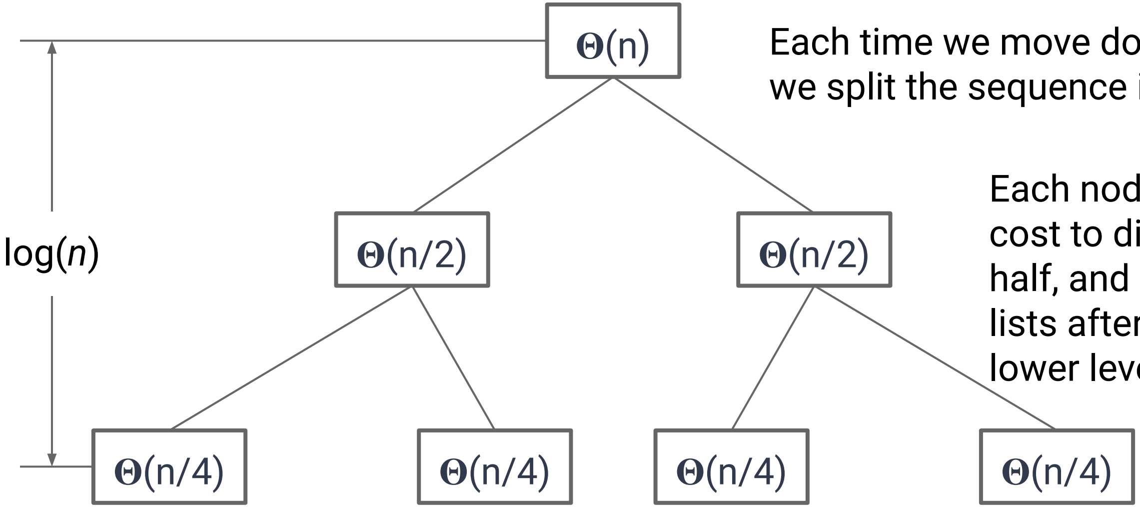

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Notice the total cost of each level is always $\Theta(n)$

Hypothesis: The cost of merge sort is n log(n)

Merge Sort: Proof by Induction

Base Case: $T(1) \leq c$

$c_0 \leq c$

True for any $c > c_0$

Merge Sort: Proof by Induction

Assume: $T(n/2) \leq c\,(n/2)\log(n/2)$

Show: $T(n) \leq cn\log(n)$

Merge Sort: Proof by Induction

Assume: $T(n/2) \leq c\,(n/2)\log(n/2)$

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By the assumption, and transitivity, we just need to show:

$$2c\frac{n}{2} \log\left(\frac{n}{2}\right) + c_1 + c_2 n \leq cn \log(n)$$

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$$cn \log(n) - cn \log(2) + c_1 + c_2 n \leq cn \log(n)$$

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Merge Sort: Proof by Induction

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Merge Sort: Proof by Induction

$$c_1 + c_2 n \leq cn \log(2)$$

$$\frac{c_1}{n \log(2)} + \frac{c_2}{\log(2)} \leq c$$

Merge Sort: Proof by Induction

Which is true for any

and

Merge Sort

Where is all of the "work" being done?

Merge Sort

Where is all of the "work" being done?

The combine step

Merge Sort

Where is all of the "work" being done?

The combine step

Can we put the work in the divide step instead?

QuickSort

Idea: What if we divide our sequence around a particular value?

What value would we like to choose?

QuickSort

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What value would we like to choose? **Median**

QuickSort: Idealized Version

7 1 4 3 5 2 6 8

QuickSort: Idealized Version

7 1 4 3 | 5 2 6 8

QuickSort: Idealized Version

7 1 4 3 | 5 2 6 8

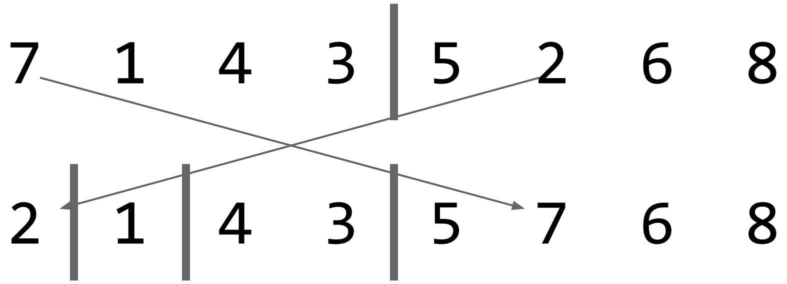
2 1 4 3 | 5 7 6 8

QuickSort: Idealized Version

7 1 4 3 | 5 2 6 8

2 1 | 4 3 | 5 7 6 8

QuickSort: Idealized Version



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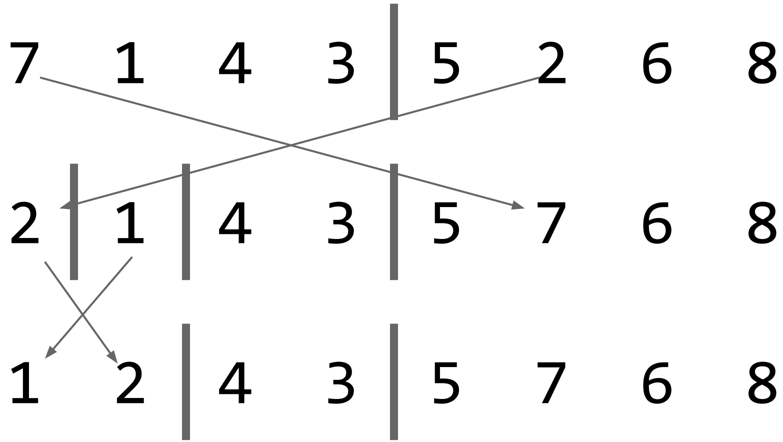

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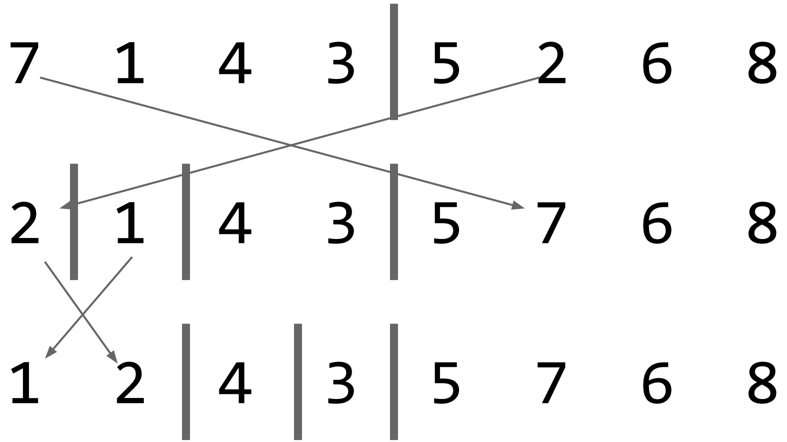

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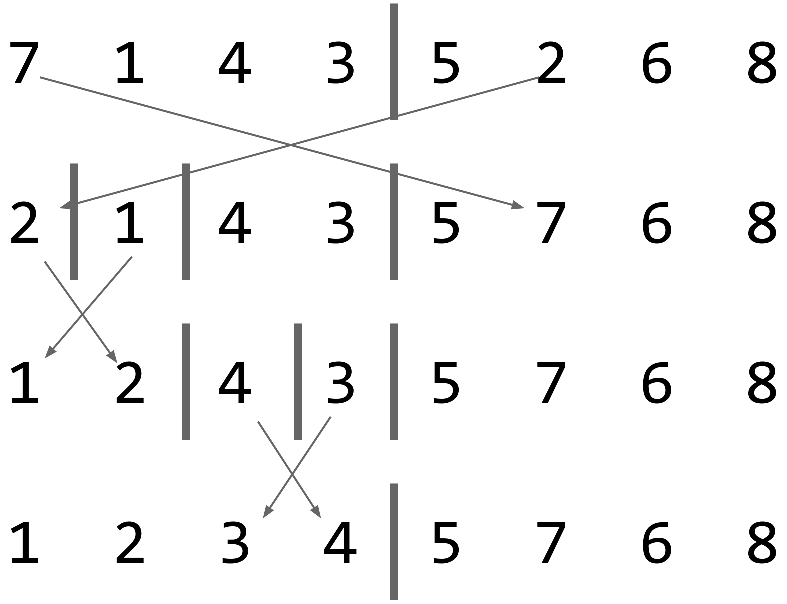

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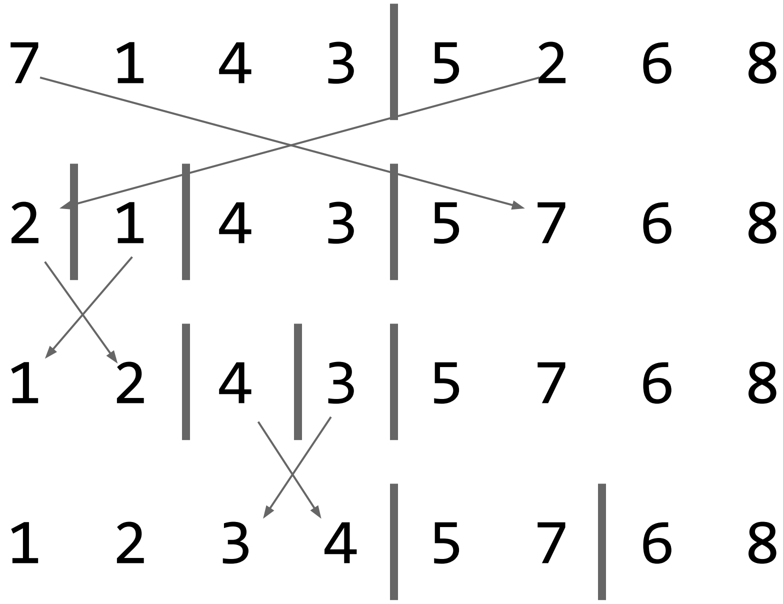

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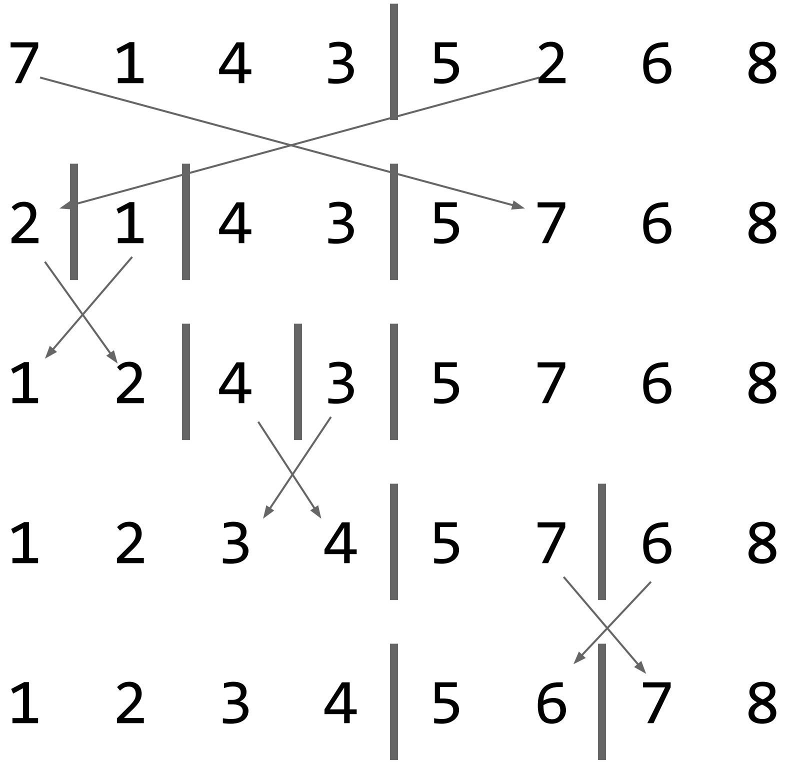

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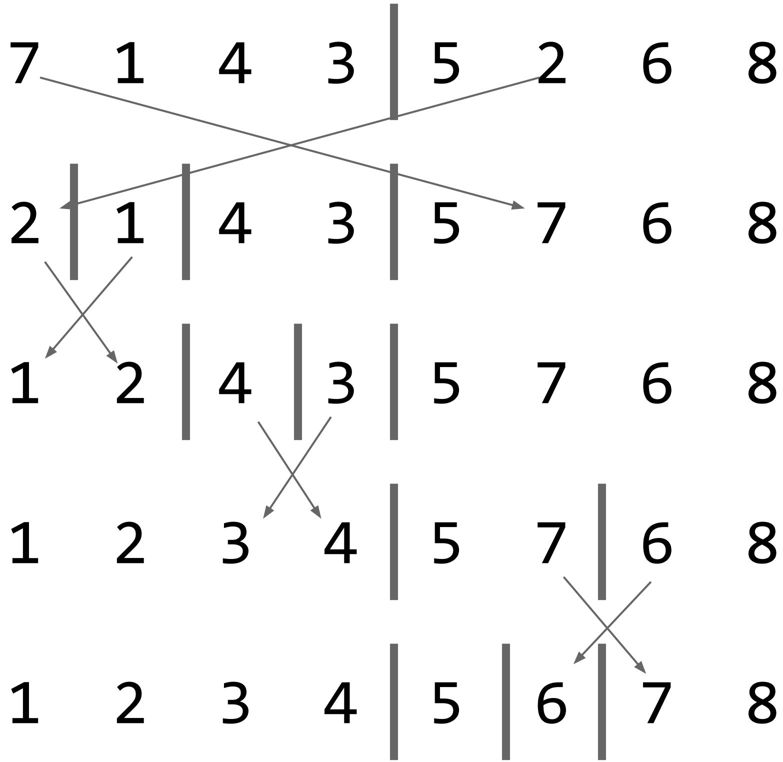

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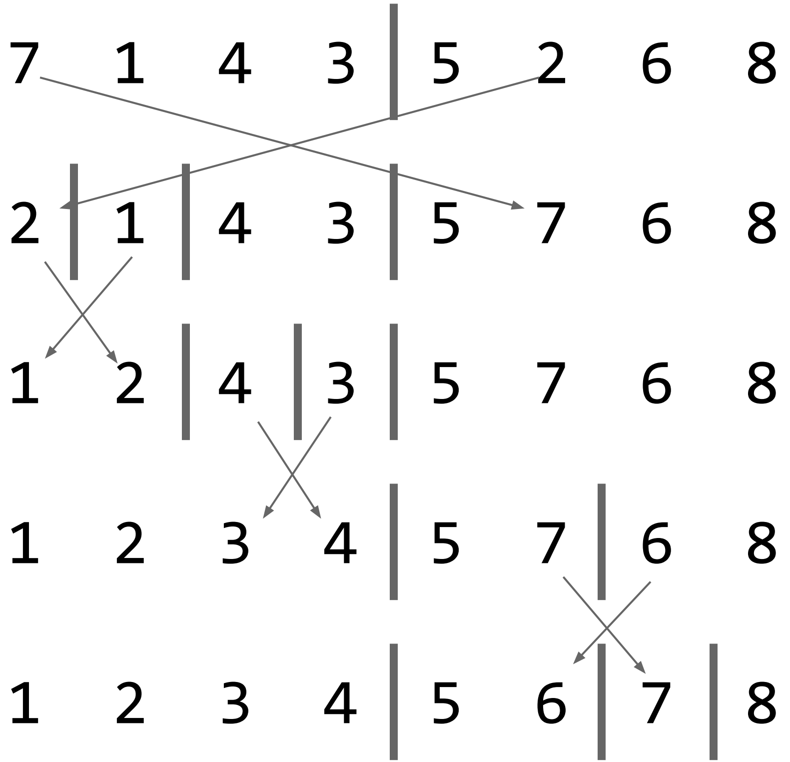

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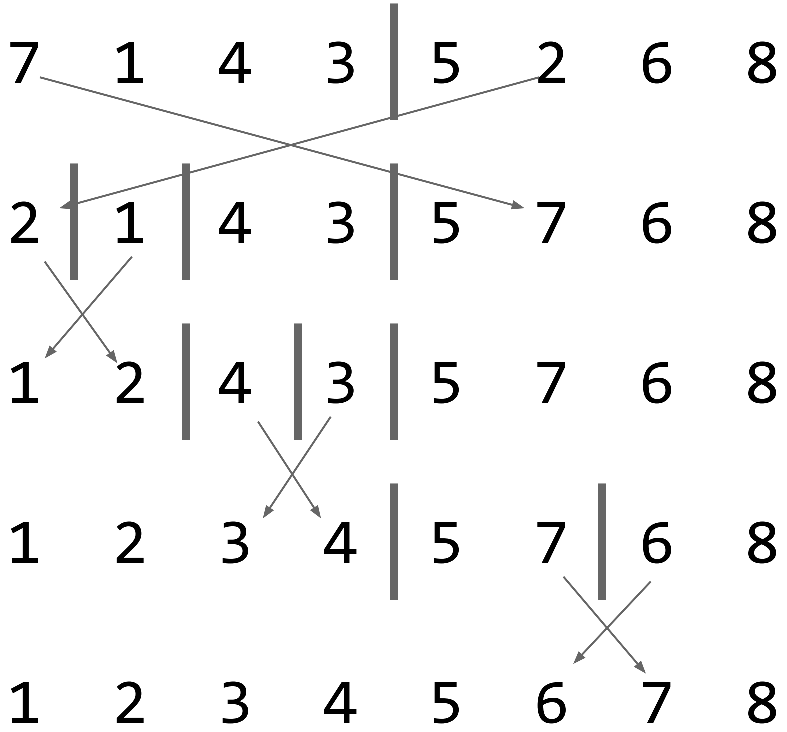

QuickSort: Idealized Algorithm

To sort an array of size *n*:

1. Pick a *pivot* value (median?)
2. Swap values until:
   a. elements at [1, *n*/2) are ≤ pivot
   b. elements at [*n*/2, *n*) are > pivot
3. Recursively sort the lower half
4. Recursively sort the upper half

QuickSort: Idealized Version

```
def idealizedQuickSort(arr: Array[Int], from: Int, until: Int): Unit = {
    if(until - from < 1) { return }
    val pivot = ???
    var low = from, high = until -1

    while(low < high) {
        while(arr(low) <= pivot && low < high){ low ++ }
        if(low < high) {
            while(arr(high) > pivot && low < high){ high ++ }
            swap(arr, low, high)
        }
    }
    idealizedQuickSort(arr, from = 0,   until = low)
    idealizedQuickSort(arr, from = low, until = until)
}
```

Great! So…how do we find the median…?

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Finding the median takes $O(n \log(n))$ for an unsorted array :(

QuickSort: Hypothetical

Imagine a world where we can obtain a pivot in $O(1)$.

Now what is our complexity?

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Now what is our complexity?

$$T_{quicksort}(n) = \begin{cases} \Theta(1) & \textbf{if } n = 1 \\ 2 \cdot T(\frac{n}{2}) + \Theta(n) + 0 & \textbf{otherwise} \end{cases}$$

QuickSort: Hypothetical

Imagine a world where we can obtain a pivot in *O*(1).

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$$T_{quicksort}(n) = \begin{cases} \Theta(1) & \textbf{if } n = 1 \\ 2 \cdot T(\frac{n}{2}) + \Theta(n) + 0 & \textbf{otherwise} \end{cases}$$

Compare to Merge Sort:

$$T_{mergesort}(n) = \begin{cases} \Theta(1) & \textbf{if } n = 1 \\ 2 \cdot T(\frac{n}{2}) + \Theta(1) + \Theta(n) & \textbf{otherwise} \end{cases}$$

QuickSort: Attempt #2

So how can we pick a pivot value (in $O(1)$ time)?

QuickSort: Attempt #2

So how can we pick a pivot value (in $O(1)$ time)?

Idea: Pick it randomly! On average, half the values will be lower.

QuickSort: Attempt #2

To sort an array of size *n*:

1. Pick a value at random as the *pivot*
2. Swap values until the array is subdivided into:
    a. *low*: array elements < *pivot*
    b. *pivot*
    c. *high:* array elements > pivot
3. Recursively sort *low*
4. Recursively sort *high*

QuickSort: Runtime

What is the worst-case runtime?

QuickSort: Worst-Case Scenario

What if we always pick the worst pivot?

```
[8,7,6,5,4,3,2,1]
```

QuickSort: Worst-Case Scenario

What if we always pick the worst pivot?

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[8,7,6,5,4,3,2,1]
```

```
[7,6,5,4,3,2,1],8,[]
```

QuickSort: Worst-Case Scenario

What if we always pick the worst pivot?

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[8,7,6,5,4,3,2,1]

[7,6,5,4,3,2,1],8,[]

[6,5,4,3,2,1],7,[],8
```

QuickSort: Worst-Case Scenario

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[5,4,3,2,1],6,[],7,8
```

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[5,4,3,2,1],6,[],7,8

...
```

QuickSort: Worst-Case Runtime

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$$T_{quicksort}(n) \in O(n^2)$$

QuickSort: Worst-Case Runtime

Is the worst case runtime representative?

QuickSort: Worst-Case Runtime

Is the worst case runtime representative?

No! (the actual runtime will almost always be faster)

QuickSort: Worst-Case Runtime

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No! (the actual runtime will almost always be faster)

But what **can** we say about runtime?

QuickSort

Let's say we pick Xth largest element for our pivot.

What is the runtime ($T(n)$)?

QuickSort

Let's say we pick Xth largest element for our pivot.

What is the runtime ($T(n)$)?

$$
\begin{cases}
T(0) + T(n-1) + \Theta(n) & \textbf{if } X = 1 \\
T(1) + T(n-2) + \Theta(n) & \textbf{if } X = 2 \\
T(2) + T(n-3) + \Theta(n) & \textbf{if } X = 3 \\
.. & \\
T(n-2) + T(1) + \Theta(n) & \textbf{if } X = n-1 \\
T(n-1) + T(0) + \Theta(n) & \textbf{if } X = n
\end{cases}
$$

Probabilities

How likely are we to pick $X = k$ for any specific k?

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$$P[X = k] = 1/n$$

Probability Theory (Great Class...)

If I roll a d6 (6-sided die) *k* times,

what is the average roll over all possible outcomes?

k = 1

If I rolld a d6 1 time…

Roll	Probability	Outcome
⚀	1/6	1
⚁	1/6	2
⚂	1/6	3
⚃	1/6	4
⚄	1/6	5
⚅	1/6	6

Expected Runtime

Back to Induction

Hypothesis: $E[T(n)] \in O(n \log(n))$

Base Case

Base Case: $E[T(1)] \leq c\ (1 \log(1))$

Base Case

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$$E[T(1)] \leq c\ (1 \cdot 0)$$

Base Case

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$$E[T(1)] \leq c\ (1 \cdot 0)$$

$$E[T(1)] \nleq 0$$

Base Case (Take 2)

Base Case (Take Two): $E[T(2)] \le c\ (2 \log(2))$

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$$T(0) + T(1) + 2c_1 \leq 2c$$

$$2c_0 + 2c_1 \leq 2c$$

True for any $c \geq c_0 + c_1$

Inductive Case

Assume: $E[T(n')] \leq c\,(n' \log(n'))$ for **all** $n' < n$

Show: $E[T(n)] \leq c\,(n \log(n))$

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$$\frac{2}{n}\left(\sum_{i=0}^{n-1} E[T(i)]\right) + c_1 \leq cn\log(n)$$

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$$c\frac{2}{n}\left(\sum_{i=0}^{n-1} i\log(n)\right) + c_1 \leq cn\log(n)$$

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$$cn\log(n) - c\log(n) + c_1 \leq cn\log(n)$$

$$c_1 \leq c\log(n)$$

QuickSort

So…is QuickSort $O(n \log(n))$...?

No!

What guarantees do you get?

If $f(n)$ is a Tight Bound

The algorithm always runs in $cf(n)$ steps

If $f(n)$ is a Worst-Case Bound

The algorithm always runs in at most $cf(n)$

If $f(n)$ is an Amortized Worst-Case Bound

n invocations of the algorithm **always** run in $cnf(n)$ steps

If $f(n)$ is an Average Bound

…we don't have any guarantees